Time reversal transformation in stochastic dynamics

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A time reversal transformation for a system of stochastic differential equations is defined. The stochastic integration is considered in the generalized Stratonovich prescription.[b]

Celebrating 45 years of the Theoretical Physics Department of the Institute of Cybernetics, Mathematics and Physics (ICIMAF), with those who have contributed to its growth, was a special moment and a great idea. We are not allowed to simply perform a time reversion and live a moment again, but, instead, we have important memories to share and common points in our academic lives.

Beyond human dynamics, time reversal symmetry is a central condition for the study of equilibrium in physical systems. On the other hand, stochastic dynamics provide an interesting approach to out-of-equilibrium statistical mechanics. The understanding of stochastic dynamics and the evolution to equilibrium for systems dealing with multiplicative noise continues to be a challenge and it is a subject of continuous and current scientific research. In this area, Langevin and Fokker-Planck formalisms are extensively used. Typically, a Langevin equation describes the motion of a diffusive particle in a medium which is modeled by splitting its effects into two parts: a deterministic part, given by a viscous force, and a stochastic part, given by a random force with a zero expectation value. For a non-homogeneous diffusion, fluctuations depend on the state of the system, defining a *multiplicative* stochastic process. To correctly define the Langevin equation, must be specified a particular prescription for performing the stochastic integration. The Fokker-Planck equation is an equivalent formalism, which corresponds to a partial differential equation for the time-dependent probability density. In particular, for the multiplicative noise case, the Fokker-Planck equation does depend on the chosen prescription for the stochastic integration of the associated Langevin equation.

A persistent problem in this area is given by the number of conventions available for performing the stochastic integration and, furthermore, by the uncertainties associated to the natural integration of a given system. In this work, we use the *generalized Stratonovich* [4] or *α – prescription* [5], where α is defined as a continuous parameter, $0 \leq \alpha \leq 1$, and each of its values corresponds to a different discretization rule for the stochastic differential equation. It is convenient to formulate a prescription-independent formalism as long as some properties, like time reversibility, mix different prescriptions. In this work, the time reversal transformation for one-variable dynamics defined in [1, 2] is generalized to the case of a system of stochastic differential equations [3], pointing out some important specific issues. This research topic is developed in collaboration with Prof. Daniel Gustavo Barci, from the Theoretical Physics Department at UERJ[c].

We consider the system of Langevin equations

$$\frac{dx_i(t)}{dt} = f_i(\mathbf{x}(t)) + g_{ij}(\mathbf{x}(t))\eta_j(t) \tag{1}$$

where $i = 1, \ldots, n$, $j = 1, \ldots, m$, $\mathbf{x} \in \mathbb{R}^n$ and $\eta_j(t)$ are m independent Gaussian white noises. We use bold face characters for vector variables and summation over repeated indices is understood. The drift force $f_i(\mathbf{x})$ and the diffusion matrix $g_{ij}(\mathbf{x})$ are, in principle, arbitrary smooth functions of $\mathbf{x}(t)$. For defining the stochastic integration in Eq. (1), we use the generalized convention

$$g_{ij}(x(\tau_k)) = g_{ij}(\alpha x(t_k) + (1-\alpha)x(t_{k-1})) \ , \tag{2}$$

with $0 \leq \alpha \leq 1$. In this way, $\alpha = 0$ corresponds with the pre-point Itô interpretation and $\alpha = 1/2$ coincides with the Stratonovich one. The post-point prescription, $\alpha = 1$, is known as the kinetic interpretation. In this context, we developed a path integral formalism, and discussed the concept of time reversibility for one-variable multiplicative noise stochastic processes, deducing a specific time reversal transformation which mixes different prescriptions [1, 2].

Any stochastic process described by a SDE in the α-prescription, can also be described by *another SDE* in any other prescription β. To be precise, suppose that $\mathbf{x}(t)$ is a solution of the system (1), where the Wiener integral is understood in the α-prescription. Then, $\mathbf{x}(t)$ is also a solution of

$$\frac{dx_i}{dt} = [f_i(\mathbf{x}) + (\alpha - \beta)g_{kj}(\mathbf{x})\partial_k g_{ij}(\mathbf{x})] + g_{ij}(\mathbf{x})\eta_j \ , \tag{3}$$

interpreted in the β-prescription. Therefore, we can represent the *same stochastic process* in different prescriptions by just shifting the drift through

$$f_i(\mathbf{x}) \to f_i(\mathbf{x}) + (\alpha - \beta)g_{k\ell}(\mathbf{x})\partial_k g_{i\ell}(\mathbf{x}). \tag{4}$$

A detailed demonstration of Eq. (4) can be found in [3].

An important observation is that different values of α imply different calculus rules. It can be summarized through the "chain rule" in the α-prescription. Consider, for instance, an arbitrary function of the stochastic variable $\mathbf{x}(t)$ satisfying the Eq. (1) in the α-prescription. Then,

$$\frac{dF(\mathbf{x}(t))}{dt} = \partial_k F \frac{dx_k}{dt} + \left(\frac{1-2\alpha}{2}\right) g_{ik} g_{jk} \partial_i \partial_j F. \quad (5)$$

In a multiplicative process driven by white noise, time reversal is a very subtle issue since the definition of backward trajectories depends on the stochastic prescription α. It is simple to see that, due to the stochastic integration particularities, the backward time evolution of $\mathbf{x}(t)$ in Eqs. (1) cannot be obtained by just changing the sign on the velocity $d\mathbf{x}/dt \to -d\mathbf{x}/dt$. Carefully analyzing the forward and backward evolution for a time interval $(t, t+\Delta t)$ in the simplest case of $f_i(\mathbf{x}) = 0$, the time reversal integral is obtained by changing α by $(1-\alpha)$.

So, the time reversed stochastic evolution is characterized by the transformations $\mathbf{x}(t) \to \mathbf{x}(-t)$ and $\alpha \to (1-\alpha)$. In this sense, we say that the prescription $(1-\alpha)$ is the time reversal conjugate of α. Then, the post-point interpretation (also known as anti-Itô prescription) is the time reversal conjugate of the pre-point Itô one, and vice versa. The only *time reversal invariant prescription* is the Stratonovich one, $\alpha = 1/2$. This means that, except for the Stratonovich case, the backward and forward stochastic paths do not have the same end points. Consider that we want to compute the evolution of the system starting at $\mathbf{x}(t)$, going forward a time interval Δt and then, turning back in time the same interval $-\Delta t$. Using the appropriate formulation and considering a very small Δt, we find

$$\Delta_\alpha x_i(t) = (2\alpha - 1) g_{k\ell}(\mathbf{x}) \partial_k g_{i\ell}(\mathbf{x}) \Delta t, \quad (6)$$

for the difference of these stochastic paths. So, in general, forward and backward time evolutions do not have the same endpoints. This also implies that forward and backward time evolutions cannot be described by the same drift function.

This point will be addressed through the Fokker-Planck formalism. The time dependent probability $P(\mathbf{x}, t)$ for the vector variable $\mathbf{x}$ satisfies

$$\frac{\partial P(\mathbf{x}, t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{x}, t) = 0, \quad (7)$$

where the current $\mathbf{J}(\mathbf{x}, t)$ is given by

$$J_i(\mathbf{x}, t) = [f_i + \alpha g_{k\ell} \partial_k g_{i\ell}] P(\mathbf{x}, t) - \frac{1}{2} \partial_j [g_{i\ell} g_{j\ell} P(\mathbf{x}, t)]. \quad (8)$$

We expect that, at very long times, the probability converges to a stationary state

$$P_{\rm st}(\mathbf{x}) = \lim_{t \to +\infty} P(\mathbf{x}, t), \quad (9)$$

for which, $\nabla \cdot \mathbf{J}_{\rm st}(\mathbf{x}) = 0$. We can write the Fokker-Planck equation for a backward trajectory $\hat{P}(\mathbf{x}, t)$ by replacing $t \to -t$, $\alpha \to 1-\alpha$ and $\mathbf{f} \to \hat{\mathbf{f}}$. The stationary states of both equations should be related by

$$P_{\rm st, [J_{st}]}(\mathbf{x}) = \hat{P}_{\rm st, [-J_{st}]}(\mathbf{x}). \quad (10)$$

That is, for an equilibrium state ($J_{st} = 0$), the asymptotic forward and backward probabilities should be the same. Looking for the backward stochastic process that fulfills this requirement, we obtain

$$\hat{f}_i(\mathbf{x}) = f_i(\mathbf{x}) + (2\alpha - 1)\; g_{k\ell}(\mathbf{x}) \partial_k g_{i\ell}(\mathbf{x}). \quad (11)$$

Finally, the time reversal transformation $\mathcal{T}$ which makes physical sense, meaning that it produces a backward evolution which converges to a unique equilibrium distribution, is given by

$$\mathcal{T} = \begin{cases} \mathbf{x}(t) & \to \quad \mathbf{x}(-t) \\ \alpha & \to \quad (1-\alpha) \\ f_i & \to \quad f_i + (2\alpha - 1)\; g_{k\ell} \partial_k g_{i\ell} \end{cases} \quad (12)$$

Conclusions

We have discussed general multidimensional stochastic processes driven by a system of Langevin equations with multiplicative noise. We have addressed the problem arising from the variety of conventions available to deal with stochastic integrals and how time reversal diffusion processes are affected by such conventions. In order to have a unique equilibrium distribution, the definition of time-reversed stochastic process is given by a specific transformation, where we need to change not only the velocity sign, but also the prescription and the drift force.

Notes

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b. Original version of this article is Ref. [3]

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