Anisotropic magnetized compact stars: the γ metric model

Diana Alvear Terrero[a1], Samantha López Pérez[1], Daryel Manreza Paret[2], Aurora Pérez Martínez[1], and Gretel Quintero Angulo[2]

[1]Departamento de Física Teórica, Instituto de Cibernética, Matemática y Física, Calle E esq. 15 No. 309, La Habana, CP 10400, Cuba
[2]Departamento de Física Teórica, Facultad de Física, Universidad de la Habana, San Lázaro y L, Vedado, La Habana 10400, Cuba

When modeling the structure of compact stars, the presence of magnetic fields poses a number of questions about the micro and macrophysics treatment of the system. Here, we discuss a model based in the so-called γ-metric to describe anisotropic magnetized compact stars.

In astrophysics, the term compact star refers to the final stage of the life of a star. Depending on the mass of the progenitor, it can be classified into three main classes: white dwarf (WD), neutron star (NS), and black hole. Degenerate stars –WDs and NSs– are made up of nuclei immerse in a sea of degenerate matter, whose pressure supports the star against the inward pull of gravity. In this paper, we will use the terms compact and degenerate stars indistinctly.

To model compact stars, there are two key ingredients to consider. One is the equation of state (EoS), whose task is to describe the thermodynamic properties specifying the state of the matter composing the star, in particular, the dependence of the pressure and energy density on particle density. The other ingredient are the structure equations, which define the variation of the pressure, mass and luminosity throughout the different layers of the star.

The structure of an object in hydrostatic equilibrium, subject only to gravity, can be simply described by Euler and Poisson equations in the Newtonian approximation. As occurs for NSs, general relativity corrections must be introduced if the matter density produces an appreciable space-time curvature, i.e. $GM/R \gg 1$, with M and R the mass and radius of the star and G the gravitational constant. In this case, the assumption of a static, spherically symmetric distribution of matter in chemical, hydrostatic and thermodynamic equilibrium leads to the Tolman-Oppenheimer-Volkoff (TOV) equations [1, 2]. The solution of this system of differential equations together with the EoS provides the spatial distribution of energy density $\epsilon(r)$, mass $M(r)$ and pressure $P(r)$. To obtain it, we integrate from $P(0) = P_c$ and $M(0) = 0$ until the pressure of the gas becomes zero, $P(R) = 0$, which determines the radius R of the star. The EoS input is used microscopically at each point of integration. It establishes the central pressures and energy densities, and their subsequent values, thus defining a parametric family of M vs R curves –the mass-radius relation–.

However, if there are magnetic fields present, combining both ingredients is not straightforward. First and foremost, the thermodynamics of a quantum system under the action of a magnetic field changes: the transverse momentum of the particles couples to the magnetic field, which results in the splitting of the pressure into two components, one parallel ($P_{\parallel}$) and the other perpendicular ($P_{\perp}$) to the magnetic field [3]. Such anisotropy contradicts the assumption of a spherically symmetric distribution of matter and an isotropic energy-momentum tensor that led to the TOV equations. Therefore, some questions arise regarding which pressure to use in the calculations and how to introduce the full thermodynamic information of the system consistently. After different attempts to address these issues, we constructed a general model suitable to study the structure of axially deformed magnetized compact stars [4, 6, 5]. We based our work in a number of papers showing that an object with axial symmetry can be described by the so-called γ-metric, a static, axisymmetric, asymptotically flat family of solutions to the Einstein equations in spherical coordinates. This metric exhibits a parameter $\gamma = z/r$ that associates the polar (z) and equatorial (r) radii and is therefore connected to the shape of the object. In the case of small deformations, $\gamma \simeq 1$, the metric can be written as

$$ds^2 = -\left[1 - 2m(r)/r\right]^{\gamma} dt^2 + \left[1 - 2m(r)/r\right]^{-\gamma} dr^2 + r^2 \sin\theta d\phi^2 + r^2 d\theta^2 \quad (1)$$

The second parameter, m, is related to both the gravitational mass $M = \gamma m$, and to the quadrupolar moment $Q = m^3\gamma(1-\gamma^2)/3$. Notice that when $\gamma \to 0$ the metric reduces to Minkowski ($M = Q = 0$) and if $\gamma \to 1$ it becomes Schwarszchild ($Q = 0$). From Eq. 1 and with an isotropic energy-momentum tensor, the the pressure follows the differential equation

$$\frac{dP}{dr} = -\frac{(E+P)\left[\frac{r}{2} + 4\pi r^3 P - \frac{r}{2}\left(1 - \frac{2M}{r}\right)^{\gamma}\right]}{r^2\left(1 - \frac{2M}{r}\right)^{\gamma}}. \quad (2)$$

Since it is our interest to study the effects of the anisotropy coming from the magnetic field on the structure equations, let us analyse the dependence of the pressures with the coordinates. The parallel pressure, directed along the z-axis, achieves its maximum value in the equatorial plane and becomes zero at the surface. Thus we assume $P_\parallel = P_\parallel(z(r))$. On the contrary, the perpendicular pressure is directed transversal to the z-axis, so we consider $P_\perp = P_\perp(r)$. That settled, we ponder the fact that when solving TOV equations, smaller pressures lead to smaller radii. Intuitively, we could think that the ratio of the pressures should be related to the shape of the object. Hence, the main supposition of our model is the ansatz of interpreting γ as the ratio between the parallel and perpendicular central pressures, that is, $\gamma = z/r = P_\parallel(z(r))/P_\perp(r) \equiv P_{\parallel 0}/P_{\perp 0}$.

At this point, we have expressions for each pressure and their dependence with the coordinates as well as the value of γ from the EoS and we are only missing how to obtain the mass. Yet, the deformed compact star may be visualized as a spheroidal object and the mass can be computed consequently. At the end, we have the following set of coupled structure equations

$$\frac{dM}{dr} = 4\pi\gamma r^2 \frac{(E_\parallel + E_\perp)}{2}, \tag{3}$$

$$\frac{dP_\perp}{dr} = -\frac{(E_\perp + P_\perp)[\frac{r}{2} + 4\pi r^3 P_\perp - \frac{r}{2}(1-\frac{2M}{r})^\gamma]}{r^2(1-\frac{2M}{r})^\gamma}, \tag{4}$$

$$\begin{aligned} \frac{dP_\parallel}{dz} &= \frac{1}{\gamma}\frac{dP_\parallel}{dr} \\ &= -\frac{(E_\parallel + P_\parallel)[\frac{r}{2} + 4\pi r^3 P_\parallel - \frac{r}{2}(1-\frac{2M}{r})^\gamma]}{\gamma r^2(1-\frac{2M}{r})^\gamma}, \end{aligned} \tag{5}$$

where $M(r)$ is the total mass enclosed in the spheroid of equatorial radius r, and at each integration step $E_\parallel = E(P_\parallel), E_\perp = E(P_\perp)$ are computed through the parametric dependence of the energy density with each pressure introduced by the EoS.

In general terms, Eqs. (3-5) are solved similarly to the TOV equations. The initial conditions $E_0 = E(r = 0)$, $P_{\parallel_0} = P_\parallel(r = 0)$ and $P_{\perp_0} = P_\perp(r = 0)$ are taken from the EoS, while the condition $P(R) = 0$ defines the equatorial radius, so that the polar radius is $Z = \gamma R$ and the total mass $M = M(R)$.

There are two important features and strengths of this model. First, as a result of using not one but both pressures, we are able to include all of the microphysics information in the EoS. The second is the fact that as expected, the system reduces to TOV equations when the magnetic field is equal to zero, i.e. for $\gamma = 1$ and $P_\parallel = P_\perp$.

However, there is still room for improvement. As a result of neglecting the dependence on the angular variables and assuming that $P_\perp$ evolves in the equatorial direction and $P_\parallel$ in the polar one, we ought to use different values of the energy density to integrate Eqs. (4) and (5). In consequence, to compute the mass of the spheroidal object the average energy density must be considered. This warns us that a complete description of the anisotropic object should stem from a full tridimensional treatment. Another limitation lies within the ansatz, since reducing the value of γ to the ratio of the central pressures dismisses the possible variation of γ throughout the different layers of the star. A more accurate solution could be achieved in the latter case.

A few years of testing the model have shown its usefulness and practicality for the study of WDs, Bose-Einstein condensate stars and Strange quark stars (see Refs. [4, 6, 5]). Common results evidence that the deformation is small in all cases, becoming relevant at the low-intermediate density regime with respect to the magnetic field, when the magnetic force starts to compare to the matter pressure. Also, the maximum masses are not significantly affected if at all, confirming that the main effect is the change in shape of the star to an oblate (prolate) spheroid depending on whether $\gamma < 1$ and $P_\parallel < P_\perp$ ($\gamma > 1$, $P_\parallel > P_\perp$), see Figure 1. To what extent this is fixed by the assumptions of the model should be further studied.

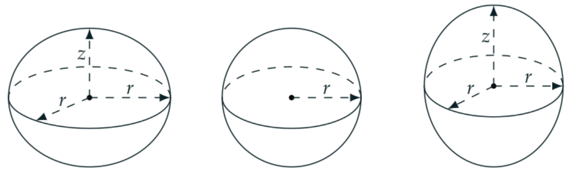

Figure 1: Scheme of deformation in γ metric model.

Notes

a. Email: dianaalvear@icimaf.cu

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