Theoretical Analysis of the State Space of Fuzzy Cognitive Maps using Shrink Functions

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We proposed definitions and theorems regarding Fuzzy Cognitive Maps (FCMs), which allow estimating bounds for the activation value of each neuron and analyzing the covering and proximity of feasible activation spaces. The main theoretical findings suggest that the state space of any FCM model equipped with transfer F-functions shrinks infinitely with no guarantee for the FCM to converge to a fixed point but to its limit state space. This result, in conjunction with the covering and proximity values of FCM-based models, helps to understand their poor performance when solving complex simulation problems.

Introduction

Fuzzy Cognitive Maps (FCMs) [1] are recurrent neural networks for modeling complex systems. Existing theoretical studies on FCMs are mainly devoted to convergence issues, commonly covering the existence and uniqueness of fixed points [2, 3]. Other results reported in [4, 5, 6] address the convergence of FCM models used in prediction/classification scenarios.

Concerning the theoretical analysis of FCMs' dynamics, we summarize our paper *Unveiling the Dynamic Behavior of Fuzzy Cognitive Maps* [7]. First, we introduce several definitions and theorems that allow studying the dynamic behavior of FCMs equipped with monotonically increasing functions bounded into non-negative intervals. The strong version of our theorem proves that the state space of an FCM shrinks infinitely and converges to a so-called *limit state space*. This allows envisaging the FCM model's behavior before the inference stage. As a second contribution, we explore the covering and proximity of *feasible activation spaces*, which help explain why FCMs sometimes perform poorly when solving complex prediction problems. In other words, we should not expect impressive prediction rates when the model has low covering values, as the FCM feasible state space is small.

Shrink Functions and State Space Estimation in FCM-based Models

We define F as the set of all monotonically increasing functions bounded into non-negative intervals. Also, let $f_i \in F$ be the transfer function used in the activation process of neuron C_i in the FCM. In [7], we refer to an F-function as any function belonging to F.

Let $\mathcal{H}_W$ and $\mathcal{H}_T$ be functions that take an FCM-based model $\mathcal{M}$ and a feasible state space at the t-th iteration $\mathcal{S}^{(t)}$ for this map and return a feasible state space at the $(t+1)$-th iteration $\mathcal{S}^{(t+1)}$ for the same map. While $\mathcal{H}_W$ uses the weight matrix W of $\mathcal{M}$ to calculate a feasible state space for the $(t+1)$-th iteration, $\mathcal{H}_T$ uses the FCM's topology only. Based upon estimated bounds for the successive activation values and from the monotonically increasing property of $f_i \in F$, we assert that over the same FCM, these two shrink functions transform feasible state spaces into state spaces which are also feasible.

To show that FCMs are not completely unpredictable, we propose two theorems as the pillars of our state-space estimation: the *Weak Shrinking State Space* and the *Strong Shrinking State Space*. The former asserts that the state spaces shrink from one iteration to the next one, although it is possible that $\mathcal{S}^{(t)} = \mathcal{S}^{(t+1)}$, which would imply that $\mathcal{S}^{(t)} = \mathcal{S}^{(t+k)}\ \forall k \in \mathbb{N}$. So, the state spaces may not shrink forever. The latter only varies in the sense that transfer functions are now bounded into open intervals. This means that the state space bounds are never reachable and hence, the state spaces will shrink forever and they will have a limit. The *limit state space* of $\mathcal{M}$ is $\mathcal{S}^{(\infty)} = \lim_{t\to\infty} \mathcal{S}^{(t)}$, when state spaces are iteratively calculated using either shrink function $\mathcal{H}_T$ or $\mathcal{H}_W$. According to simulations, $\mathcal{S}^{(\infty)}$ often contains a single point.

Covering and Proximity of FCM Models

In this section, we discuss two evaluation measures that help understand the properties of FCM-based systems. The *covering* quantifies the proportion of the induced activation space that is reachable by the neuron's activation values and the *proximity* measures the mean relative distance of neuron's activation values to the feasible activation spaces.

Small covering values are evidence of the reduced representativeness of induced activation space, but sometimes we desire high covering values to represent the most diverse sets of outputs. As illustrated, such measures have a straightforward connection with the *Strong Shrinking State Space Theorem*. More importantly, they help explain why FCMs sometimes perform poorly when applied to prediction problems that demand high accuracy.

Experimental Scenarios

For experimentation purposes, we generated 400 FCM-based models (200 stable and 200 unstable) with varied properties according to the number of neurons (5 to 30), weights ($[-1, 1]$ interval) and connectivity or percentage of relationships (10%, 20%, ..., 100%). The simulations reported more valuable results in the presence of stable FCM models and when the knowledge comprised into the weight set is available. Therefore, Figures 1, 2 and 3 correspond to this situation.

Figure 1 depicts the covering values resulting for this scenario. Higher connectivity values and higher number of map neurons have a considerable influence on attaining higher covering values.

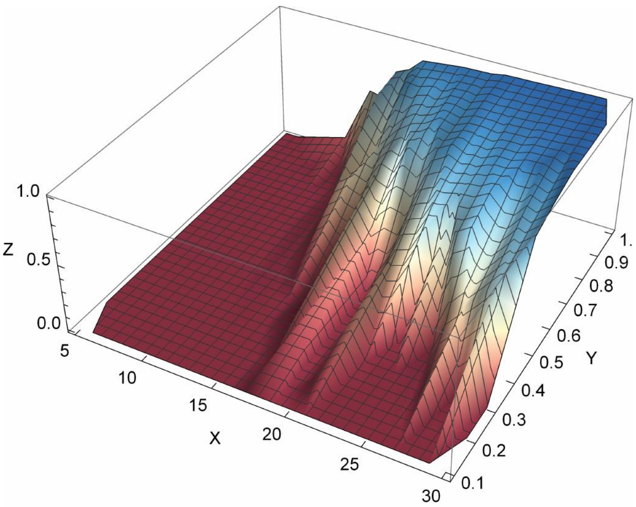

Figure 1: Axes X, Y and Z respectively refer to the number of neurons, connectivity and covering values.

A zero covering value is a computational evidence of fixed-point attractors for every one of these FCM models. According to Figure 2, at least 81 FCM model will always converge to a fixed-point attractor regardless of the initial stimulus. Moreover, in Figure 3 we can observe that nearly half of the proximity values are exactly zero and then, almost surely, 90 FCM models converge to a fixed-point attractor.

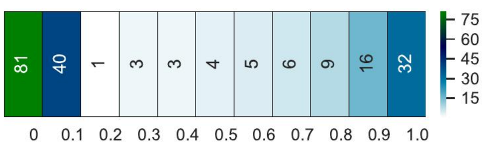

Figure 2: Distribution of covering values.

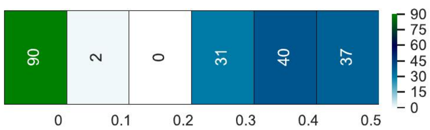

Figure 3: Distribution of proximity values.

Concluding Remarks

Our research in [7] goes a step beyond the study of fixed-point attractors, since we analyze the dynamical behavior of FCM-based models from the perspective of their state spaces. The *Strong Shrinking State Space Theorem* enunciated in this paper ensures that the feasible state space of the targeted FCMs shrinks infinitely, yet the system converges to its limit state space. As shown in the experiments, approximating an FCM's limit state space is useful to predict fixed-point attractors. Likewise, we illustrated that the covering of feasible activation spaces is often poor and irregular for FCMs with reduced network topologies. This knowledge could be injected into the learning procedure in order to improve network's performance.

Notes

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References

[1] Kosko, B. (1986). Fuzzy Cognitive Maps, *International Journal Man-Machine Studies,* **24** 1 65–75

[2] Boutalis, Y., Kottas, T., & Christodoulou, M. (2009). Adaptive Estimation of Fuzzy Cognitive Maps with Proven Stability and Parameter Convergence, *IEEE Transactions Fuzzy Systems,* **17** 4 874–889

[3] Harmati, I., Hatwágner, M., & Kóczy, L. (2018). On the Existence and Uniqueness of Fixed Points of Fuzzy Cognitive Maps, *Information Processing and Management of Uncertainty in Knowledge-Based Systems. Theory and Foundations,* 490–500

[4] Nápoles, G., Bello, R., & Vanhoof, K. (2014). How to Improve the Convergence on Sigmoid Fuzzy Cognitive Maps?, *Intelligent Data Analysis,* **18** 6S S77–S88

[5] Nápoles, G., Papageorgiou, E., Bello, R. & Vanhoof, K. (2016). On the convergence of sigmoid Fuzzy Cognitive Maps, *Information Sciences,* **349-350** 154–171

[6] Nápoles, G., Concepción, L., Falcon, R., Bello, R., & Vanhoof, K. (2018). On the accuracy-convergence tradeoff in sigmoid Fuzzy Cognitive Maps, *IEEE Transactions on Fuzzy Systems,* **26** 4 2479–2484

[7] Concepción, L., Nápoles, G., Falcon, R., Vanhoof, K., & Bello, R. (2021). Unveiling the Dynamic Behavior of Fuzzy Cognitive Maps, *IEEE Transactions on Fuzzy Systems,* **29** 5 1252-1261