SU(2)-Structure & Heterotic String Compactification

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We briefly discuss SU(2)-structure 6-manifolds, their role in the compactification of the heterotic string and the resulting gauged supergravity at low energies [1].

ΑΓΕΩΜΕΤΡΗΤΟΣ·ΜΗΔΕΙΣ·ΕΙΣΙΤΩ

String Theory replaces the idea of point-like elementary particles with that of tiny one-dimensional objects, a.k.a. strings (duh!), and attempts to describe their propagation and interaction by consistently applying relativistic and quantum principles. The resulting spin-2 vibrational state is readily identified as the graviton, and therefore String Theory is a natural candidate for a theory of quantum gravity.

Unfortunately, consistency of the theory requires a number of extra spatial dimensions. The supersymmetric string, for example, requires a ten-dimensional space-time. One solution to the discrepancy with the perceived four-dimensional world is to consider the extra six dimensions to be *compactified*, i.e. forming a very small, compact 'internal' manifold Y over every four-dimensional point.

The effective action of the superstring in four dimensions at low energies[a], obtained by integrating out the six extra dimensions, depends of course on the mathematical properties of Y. Since supersymmetry seems to solve a number of issues also at low energies, it is desirable for at least part of the 10-dimensional supersymmetry to be preserved by the compactification process. This is achieved by requiring the existence of one or more globally-defined spinors on Y, which in turn implies the reduction of the structure group of Y to a proper subgroup G of SO(6).[b]

The existence of *two* globally defined, linearly independent spinors η_1 and η_2 on Y implies SU(2)-structure and, for the heterotic string, leads to $N = 2$ supergravity in 4 dimensions. If, moreover, these spinors are covariantly constant with respect to the Levi-Civita connection, then Y has SU(2)-holonomy and therefore $Y = \mathrm{K3} \times T^2$, i.e. a product of a complex K3 surface and a torus.

SU(2)-holonomy case

The existence of such a pair of spinors, which can be chosen to satisfy the orthonormality condition $\bar{\eta}_i \eta_j = \delta_{ij}$, is equivalent to the existence of a triplet of self-dual two-forms $J^x = J^x_{ab} \mathrm{d}y^a \wedge \mathrm{d}y^b$ and a pair of real one-forms $v^i = v^i_a \mathrm{d}y^a$ on Y, as can be seen from the following relations,

$$\begin{aligned} J^1_{ab} + \mathrm{i} J^2_{ab} &= \mathrm{i}\bar{\eta}_2 \gamma_{ab} \eta_1, \\ J^3_{ab} &= -\tfrac{\mathrm{i}}{2}(\bar{\eta}_1 \gamma_{ab} \eta_1 + \bar{\eta}_2 \gamma_{ab} \eta_2), \\ v^1_a + \mathrm{i} v^2_a &= \bar{\eta}^c_2 \gamma_a \eta_1, \end{aligned} \tag{1}$$

where γ_a are the six SO(6) gamma-matrices and γ_{ab} is the antisymmetrised product $\frac{1}{2}(\gamma_a\gamma_b - \gamma_b\gamma_a)$.

For $\mathrm{K3} \times T^2$, the fact that the spinors η_i are covariantly constant with respect to the Levi-Civita connection implies that J^x and v^i as defined in Eq. (1) are closed: $\mathrm{d}J^x = \mathrm{d}v^i = 0$. In this case, J^x are the three self-dual closed forms defining the hyperkähler structure on K3, and $v^i = \mathrm{d}z^i$, with z^i the coordinates of $T^2 = S^1 \times S^1$.

The bosonic sector of the heterotic string consists of the ten-dimensional metric, the Neveu–Schwarz two-form B_2 and the $E_8 \times E_8$ Yang-Mills field A_1. An Ansatz for the Kaluza-Klein reduction on $\mathrm{K3} \times T^2$ to four dimensions x^μ can be written as follows,

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + g_{mn}\mathrm{d}y^m \mathrm{d}y^n + g_{ij}\mathcal{E}^i\mathcal{E}^j, \\ B_2 &= \tfrac{1}{2}B_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu + B_{i\mu}\mathcal{E}^i \wedge \mathrm{d}x^\mu + \\ &\qquad + \tfrac{1}{2}B_{ij}\mathcal{E}^i \wedge \mathcal{E}^j + b_A\omega^A, \\ A^I_1 &= A^I_\mu \mathrm{d}x^\mu + A^I_i \mathcal{E}^i, \end{aligned} \tag{2}$$

where $\mathcal{E}^i = \mathrm{d}z^i - V^i_\mu \mathrm{d}x^\mu$, y^m and g_{mn} are the coordinates and metric on K3, and ω^A are the 22 harmonic two-forms on K3. The original $E_8 \times E_8$ gauge symmetry will be generically broken to an Abelian subgroup $\mathrm{U}(1)^{n_\mathrm{g}}$ for some n_g, so $I = 1, \ldots, n_\mathrm{g}$.

Carrying out the Kaluza-Klein reduction with this Ansatz leads to $N = 2$ supergravity with fields organized in one gravity multiplet, $n_\mathrm{v} = 3 + n_\mathrm{g}$ vector multiplets and 20 hypermultiplets.[c] The bosonic content is distributed as follows: the metric $g_{\mu\nu}$ sits of course in the gravity multiplet; the $4 + n_\mathrm{g}$ vectors V^i_μ, $B_{i\mu}$ and A^I_μ become the vectors in the n_v vector multiplets, plus the graviphoton in the gravity multiplet; the three scalars in the symmetric g_{ij}, plus the one scalar in the antisymmetric B_{ij}, plus the dilaton ϕ, plus the scalar dual to the two-form $B_{\mu\nu}$ (the so-called axion a), plus the $2n_\mathrm{g}$ scalars A^I_i, become the n_v complex scalars in the vector multiplets; and finally, 58 scalars from the deformations of the K3 metric g_{mn}, plus the 22 scalars b^A, become the 80 real scalars sitting in the 20 hypermultiplets.

The hypermultiplet scalars can be organized in a $\mathrm{SO}(4, 20)$ matrix $\mathcal{M}$, and after tedious calculations the Lagrangian for these fields can be shown to take the following form,

$$\mathcal{L} = \tfrac{1}{8}\mathrm{tr}(\partial_\mu \mathcal{M} \partial^\mu \mathcal{M}). \tag{3}$$

In particular, no potential is generated for these or any of the scalar fields, which is seen as a problem and part of the motivation for considering a more general type of background.

General SU(2)-structure backgrounds

For a generic SU(2)-structure manifold Y, the spinors η_i are covariantly constant with respect to a connection with non-vanishing *torsion*. In this case, the forms J^x and v^i as defined in Eq. (1) fail to be closed.

The manifold Y cannot be written as a Cartesian product in this general case; however, the existence of the pair of one-forms v^i still allows for the definition of an *almost product structure* ${\Pi_a}^b = 2g^{bc}v^i_a v^i_c - \delta^b_a$ on Y satisfying ${\Pi_a}^b{\Pi_b}^c = \delta^c_a$. This structure splits the tangent space over every point y of Y into two- and four-dimensional subspaces, $T_yY = V_2 \oplus W_4$. The subspace V_2 is spanned by the vectors dual to the one-forms v^i, and the orthonormality of the spinors implies $\iota_{v^i}J^x$, i.e. the two-forms J^x must have legs only along W_4.

Due to this result, the most general departure from closure for the two- and one-forms on Y can be parametrized in the following way,[d]

$$\mathrm{d}v^i = \theta^i v^1 \wedge v^2, \quad \mathrm{d}\omega^A = T^A_{iB} v^i \wedge \omega^B, \tag{4}$$

where the number of two-forms ω^A is generically n. Vanishing torsion means $\theta^i = T^A_{iB} = 0$, $Y = \mathrm{K3} \times T^2$ and $n = 22$.

Nilpotency of the d-operator, together with Stokes' theorem $\int_Y \mathrm{d}(v^i \wedge \omega^A \wedge \omega^B) = 0$, leads to the following constraints on the values of θ^i and T^A_{iB},

$$T^A_{iB} = \theta_i \delta^A_B + \Theta^A_{iB}, \quad [\Theta_1, \Theta_2] = \theta^i \Theta_i, \tag{5}$$

where $\theta_i \equiv -\frac{1}{2}\epsilon_{ij}\theta^j$, with $\epsilon_{ij} = -\epsilon_{ji}$ and $\epsilon_{12} = 1$. Also, the two matrices ${(\Theta_i)^A}_B \equiv \Theta^A_{iB}$ are in $\mathfrak{so}(3, n-3)$, the algebra of $\mathrm{SO}(3, n-3)$, as there must be three self-dual and $n-3$ anti-self-dual two-forms.

K3 fibration over a torus

A case with $\theta^i = 0$ can be constructed as a fiber bundle with fiber K3 over a torus T^2. The matrices Θ_i can be any two mutually commuting matrices in $\mathfrak{so}(3, 19)$. The one-forms are $v^i = \mathrm{d}z^i$, as in the SU(2)-holonomy case, and the $n = 22$ two-forms ω^A are closed on every K3 fiber, but depend on z^i in the following way,

$$\omega^A(z) = {(\exp z^i \Theta_i)^A}_B \omega^B(0). \tag{6}$$

Going once around each torus coordinate ($z^i \sim z^i + 1$), the set of two-forms ω^A needs to come back to itself up to some discrete monodromy $\exp T^i$ in $\mathrm{SO}(3, 19, \mathbb{Z})$, which is indeed a symmetry of the string theory.

Performing the dimensional reduction on this background with the Ansatz in Eq. (2) produces a *gauged* $N = 2$ supergravity with the same field content. The hypermultiplet scalars become charged with respect to the Kaluza-Klein vectors V^i_μ, and the Lagrangian in Eq. (3) becomes

$$\mathcal{L} = \tfrac{1}{8}\mathrm{tr}(D_\mu \mathcal{M} D^\mu \mathcal{M}) - \tfrac{1}{8}\mathrm{e}^\phi g^{ij}\mathrm{tr}([\mathcal{M}, \mathcal{T}_i][\mathcal{M}, \mathcal{T}_j]) \tag{7}$$

with covariant derivatives

$$D_\mu \mathcal{M} = \partial_\mu \mathcal{M} - V^i_\mu [\mathcal{M}, \mathcal{T}_i], \tag{8}$$

and the 24×24 matrix $\mathcal{T}_i$ defined as

$$\mathcal{T}_i = \mathrm{diag}(0, 0, \Theta_i) \in \mathfrak{so}(4, 20). \tag{9}$$

As can be seen from Eq. (7), a potential is generated for the hypermultiplet scalars. The scalars in vector multiplets remain neutral.

K3 fibration over a 'twisted torus'

The case with $\theta^i \neq 0$ can be thought of as a K3 fibration over a 'twisted torus'. Though in all rigor a 'twisted torus' does not exist as a global manifold, the construction makes sense if we consider it in two steps: a reduction to 5 dimensions on $\mathrm{K3} \times S^1$ plus a further, Scherk-Schwarz-type compactification on another circle S^1.

The embedding into string theory remains problematic in this case, as the twist after going once around the circle is not in the U-duality group. Still, as far as the effective field theory is concerned, the result is consistent with a *gauged* $N = 2$ supergravity.

Eqs. (7) and (8) for the hypermultiplet sector still apply in this case, with

$$\mathcal{T}_i = \mathrm{diag}(\theta_i, -\theta_i, \Theta_i) \in \mathfrak{so}(4, 20). \tag{10}$$

The biggest difference is that the scalars in the vector multiplet sector become charged as well with respect to some linear combinations of the vectors V^i_μ and $B_{i\mu}$, and a potential is also generated in this sector.

The full equations for the bosonic part of the effective action are too big to fit here, but can be found in [1], together with all the necessary references.

Notes

a. 'Low' with respect to the Planck energy $\sqrt{\hbar c^5/G}$, but still quite high with respect to energies accessible to modern particle accelerators.

b. This means that one can choose orthonormal bases on an open cover of Y such that all transition functions take values in G instead of the generic SO(6) rotation.

c. There are additional hypermultiplets whose number and structure depend on the details of the breaking of the $E_8 \times E_8$ gauge symmetry, but these will be ignored here.

d. There are no other one- or three-forms if one rules out massive gravitino multiplets.

References

[1] J. Louis, D. Martinez-Pedrera and A. Micu, *JHEP* 09 (2009) 012 – arxiv.org/abs/0907.3799